INT. J. NUM. METH. HEAT FLUID FLOW, VOL. 2, 429–440 (1992)

STEADY-STATE THERMOELASTICITY BY MULTIPLE RECIPROCITY METHOD

A. C. NEVES AND A. J. NOWAK*

Wessex Institute of Technology, Ashurst Lodge, Ashurst, Southampton SO4 2AA, UK

ABSTRACT

Thermal stress problems are analysed using two coupled boundary element codes. The first code computes the temperature and heat flux distributions whereas the second one uses these results to calculate the displacements and stresses at any point. In both codes, the multiple reciprocity method is used in order to avoid the domain discretization due to the 'body force' terms. Examples with different geometry and different heat sources are presented to check the accuracy of the formulations.

KEY WORDS Multiple reciprocity method Thermoelasticity problems

INTRODUCTION

The advent of supersonic aircraft and missiles and the design of nuclear reactors have introduced into engineering the problem of finding thermoelastic stresses. Previously, some analytical solutions were available but only for simple geometries[1–3]

In this paper, two dimensional thermal stress problems in any geometry are solved using the boundary element method (BEM). Because of the complex geometry and/or boundary conditions in the thermal part of the analysis, the analytical expression for the temperature field is not available and has to be calculated numerically. Once the temperature and its normal derivatives (heat fluxes) have been found at any point, a second analysis is carried out to determine displacements and stresses at the same points.

The temperature distribution depends not only on boundary conditions but also on certain heat sources acting within the body. Because of the heat sources, the application of boundary element method leads to an integral equation which contains domain integrals. Although these integrals do not introduce any unknowns, they detract from the elegance of the formulation and affect the efficiency of the method because integrations over the whole volume are required. In order to avoid the discretization of the domain into cells[4], the multiple reciprocity method (MRM) is used. The method generates a series of boundary integrals after applying the reciprocity theorem in recurrence manner which increases the order of the fundamental solution. It was originally developed by Nowak and Brebbia to solve Poisson[5] and Helmholtz equations[6] and has been extended by Neves and Brebbia to elasticity with gravitational and centrifugal loadings[7] as well as to thermoelasticity with certain types of heat sources[8] utilizing, however, only known analytical expressions for temperature fields. As a consequence relatively simple geometries have been studied. The method has also been successfully applied in other engineering problems.

* On leave from Institute of Thermal Technology, Technical University of Silesia, Gliwice, Poland

0961–5539/92/050429–12$2.00

Received January 1992
Revised June 1992

Recently, Itagaki used it to solve one-group fission neutron problems[9]; Power and Power applied it to solve Ossen's system of equations[10]; and Kamiya and Andoh use the MRM in the analysis of Helmholtz eigenvalues[11].

The purpose of this paper is to show how the MRM can be applied to solve coupled thermoelasticity problems. It is assumed, however, that problems under consideration are linear in a sense that physical properties do not depend on solutions. Up-to-now the MRM has not been applied to solve the non-linear cases and these problems are subject of separate research. Some of the most important features as well as noticed drawbacks are pointed out. Numerical examples from the field of nuclear engineering demonstrate the accuracy of the MRM. Different heat sources and geometries are analysed and results are compared with analytical solutions and/or with another BEM approach based on integration over the domain.

ANALYSIS OF THE PROBLEM

Considering a steady-state temperature distribution, the heat conduction problem is governed by the Poisson equation given by:

$$\lambda \nabla^2 T + b = 0 \tag{1}$$

where λ is the thermal conductivity; T is the temperature distribution; b is the heat source density which depends on position only.

After applying Green's second theorem, one obtains the following integral equations for each boundary node i:

$$\lambda C_i T_i = \int_\Gamma T^* q \, d\Gamma - \int_\Gamma q^* T \, d\Gamma - \int_\Omega T^* b \, d\Omega \tag{2}$$

in which the heat flux is defined as:

$$q = -\lambda \frac{\partial T}{\partial n} \tag{3}$$

The fundamental solution T^* and the heat flux analog q^* depend on geometrical distance r and have the usual meaning:

$$T^* = \frac{1}{2\pi} \ln(r) \tag{4}$$

$$q^* = -\lambda \frac{\partial T^*}{\partial n} \tag{5}$$

The second part of the analysis is governed by the Navier equation and the resulting integral equations that describe displacements at any point and stresses at internal points[8] are given respectively by:

$$C_{ij} u_j = \int_\Gamma u^*_{ij} p_j \, d\Gamma - \int_\Gamma p^*_{ij} u_j \, d\Gamma + 2\mu \left(\frac{1+v}{1-2v} \right) \alpha \int_\Omega u^*_{ij,j} T \, d\Omega \tag{6}$$

$$\sigma_{ij} = \int_\Gamma u^*_{ijk} p_k \, d\Gamma - \int_\Gamma p^*_{ijk} u_k \, d\Gamma - 2\mu \left(\frac{1+v}{1-2v} \right) \alpha \left[\int_\Omega \sigma^*_{ij} T \, d\Omega + T \delta_{ij} \right] \tag{7}$$

where C_{ij} is the free coefficient; u_j is the displacement at any point in the j direction; p_j is the traction in the j direction; σ_{ij} is the stress tensor at internal point; α is the coefficient of linear thermal expansion; δ_{ij} is the Kronecker delta; v is the Poisson's ratio; μ is the shear modulus.

The fundamental solutions of (6) and (7) are given by:

$$u_{ij}^* = \frac{1}{8\pi(1-\nu)\mu}\{(3-4\nu)\ln(1/r)\delta_{ij} + r_{,i}r_{,j} - \tfrac{1}{2}\delta_{ij}\} \tag{8}$$

$$p_{ij}^* = \frac{-1}{4\pi(1-\nu)r}\left\{[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}]\frac{\partial r}{\partial n} - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\} \tag{9}$$

$$u_{ij,j}^* = \frac{-(1-2\nu)}{4\pi(1-\nu)}\frac{r_{,i}}{r} \tag{10}$$

$$u_{ijk}^* = \frac{1}{4\pi(1-\nu)r}\{(1-2\nu)(r_{,j}\delta_{ik} + r_{,i}\delta_{jk} - r_{,k}\delta_{ij}) + 2r_{,i}r_{,j}r_{,k}\} \tag{11}$$

$$p_{ijk}^* = \frac{\mu}{2\pi(1-\nu)r^2}\left\{2\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}] + \right.$$
$$\left. 2\nu(n_ir_{,j}r_{,k} + n_jr_{,i}r_{,k}) + (1-2\nu)(2n_kr_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\nu)n_k\delta_{ij}\right\} \tag{12}$$

$$\sigma_{ij}^* = -u_{ijk,k}^* - \frac{1}{2(1-\nu)}\delta_{ij}\Delta(\xi, x)$$
$$= \frac{(1-2\nu)}{\pi(1-\nu)}\left[\frac{1}{r^2}\left(r_{,i}r_{,j} - \frac{\delta_{ij}}{2}\right) - \frac{\pi}{2(1-2\nu)}\delta_{ij}\Delta(\xi, x)\right] \tag{13}$$

Equations (6) and (7) can only be solved when the temperature field resulting from (2) is known.

Note that the integral equations (2), (6) and (7) contain domain integrals. In order to avoid domain discretization, the multiple reciprocity method is used.

Consider first the domain integral of (2) with the upper index (0) included to indicate that this is the original expression. The application of MRM leads to a series of boundary integrals given by[5]:

$$\int_\Omega T^{*(0)}b^{(0)}\,d\Omega = \frac{1}{\lambda}\int_\Gamma \sum_{L=0}^{\infty}[T^{*(L+1)}w^{(L)} - q^{*(L+1)}b^{(L)}]\,d\Gamma \tag{14}$$

where higher order fundamental solutions and the Laplacians of source function satisfy the following group of equations:

$$\nabla^2 T^{*(L+1)} = T^{*(L)} \tag{15}$$

$$q^{*(L+1)} = -\lambda\frac{\partial T^{*(L+1)}}{\partial n} \tag{16}$$

$$b^{(L)} = \nabla^2 b^{(L-1)} \tag{17}$$

$$w^{(L)} = -\lambda\frac{\partial b^{(L)}}{\partial n} \tag{18}$$

The higher order fundamental solutions resulting from (15) and (16) are given by:

$$T^{*(L+1)} = \frac{1}{2\pi}r^{2(L+1)}[\alpha^{(L+1)}\ln(r) - \beta^{(L+1)}] \tag{19}$$

$$q^{*(L+1)} = -\frac{\lambda}{2\pi}2(L+1)r^{2L+1}\left[\alpha^{(L+1)}\left(\ln(r) + \frac{1}{2(L+1)}\right) - \beta^{(L+1)}\right]\frac{\partial r}{\partial n} \tag{20}$$

where coefficients α and β can be obtained from:

$$\alpha^{(L+1)} = \frac{\alpha^{(L)}}{4(L+1)^2} \tag{21}$$

$$\beta^{(L+1)} = \frac{1}{4(L+1)^2}\left[\frac{\alpha^{(L)}}{L+1} + \beta^{(L)}\right] \tag{22}$$

with initial values $\alpha^{(0)} = 1$ and $\beta^{(0)} = 0$.

The same procedure is applied to the domain integrals in (6) and (7) using the following recurrence formulae:

$$\nabla^2 u^{*(L)}_{ij,j} = u^{*(L-1)}_{ij,j} \tag{23}$$

$$\nabla^2 \sigma^{*(L)}_{ij} = \sigma^{*(L-1)}_{ij} \tag{24}$$

$$T^{(L)} = \nabla^2 T^{(L-1)} = -\frac{1}{\lambda} b^{(L-1)} \tag{25}$$

Since source function $b^{(0)}$ is a known function, the Laplacians in (17), and (25) can be obtained analytically.

Using the formulae (23) to (25) and the reciprocity theorem, one obtains[8]:

$$\int_\Omega u^{*(0)}_{ij,j} T^{(0)}\, d\Omega = \int_\Gamma \sum_{L=0}^{\infty} \left(\frac{\partial u^{*(L+1)}_{ij,j}}{\partial n} T^{(L)} - u^{*(L+1)}_{ij,j} \frac{\partial T^{(L)}}{\partial n}\right) d\Gamma \tag{26}$$

$$\int_\Omega \sigma^{*(0)}_{ij} T^{(0)}\, d\Omega = \int_\Gamma \sum_{L=0}^{\infty} \left(\frac{\partial \sigma^{*(L+1)}_{ij}}{\partial n} T^{(L)} - \sigma^{*(L+1)}_{ij} \frac{\partial T^{(L)}}{\partial n}\right) d\Gamma \tag{27}$$

where $T^{(0)}$ indicates the temperature itself which has to be determined by first code.

The higher order fundamental solutions in (26) and (27) are given by:

$$u^{*(L+1)}_{ij,j} = \frac{-(1-2v)}{4\pi(1-v)\mu} A^{(L+1)} r^{2L+1} r_{,i}(\ln(r) - B^{(L+1)}) \tag{28}$$

$$\frac{\partial u^{*(L+1)}_{ij,j}}{\partial n} = \frac{-(1-2v)}{4\pi(1-v)\mu} A^{(L+1)} r^{2L}[(n_i + 2L r_{,i} r_{,j} n_j)(\ln(r) - B^{(L+1)}) + r_{,i} r_{,j} n_j] \tag{29}$$

$$\sigma^{*(L+1)}_{ij} = -\frac{(1-2v)}{4\pi(1-v)} C^{(L+1)}\left[\frac{1}{1-2v}(\ln(r) - D^{(L+1)})\delta_{ij} + \left(r_{,i} r_{,j} - \frac{\delta_{ij}}{2}\right) \times (E^{(L+1)}\ln(r) - F^{(L+1)})\right] r^{2L} \tag{30}$$

$$\frac{\partial \sigma^{*(L+1)}_{ij}}{\partial n} = -\frac{(1-2v)}{4\pi(1-v)} C^{(L+1)}\left\{\frac{1}{1-2v}[2L(\ln(r) - D^{(L+1)}) + 1] r_{,k} n_k \delta_{ij} + \left[\left(r_{,i} r_{,j} - \frac{\delta_{ij}}{2}\right)(2L\ln(r) + 1) r_{,k} n_k + (n_i r_{,j} + n_j r_{,i} - 2 r_{,i} r_{,j} r_{,k} n_k)\ln(r)\right] E^{(L+1)} - \left[2L r_{,k} n_k \left(r_{,i} r_{,j} - \frac{\delta_{ij}}{2}\right) + (n_i r_{,j} + n_j r_{,i} - 2 r_{,i} r_{,j} r_{,k} n_k)\right] F^{(L+1)}\right\} r^{2L-1} \tag{31}$$

where coefficients satisfy recurrence relationships:

$$A^{(L+1)} = \frac{A^{(L)}}{(2L+1)^2 - 1} \tag{32}$$

$$B^{(L+1)} = B^{(L)} + \frac{2(2L+1)}{(2L+1)^2 - 1} \tag{33}$$

$$C^{(L+1)} = \frac{C^{(L)}}{4L^2} \tag{34}$$

$$D^{(L+1)} = D^{(L)} + \frac{1}{L} \tag{35}$$

$$E^{(L+1)} = \frac{2L}{L+1} \tag{36}$$

$$F^{(L+1)} = (LF^{(L)} + E^{(L+1)})\frac{L}{L^2 - 1} \tag{37}$$

In this case, initial values for $L = 0$ are $A^{(1)} = 1/2$, $B^{(1)} = 0$, $C^{(1)} = 1$, $D^{(1)} = 0$, $E^{(1)} = 0$ and $F^{(1)} = -1$; and for $L = 1$, $F^{(2)} = 0$.

MAIN FEATURES OF THE MULTIPLE RECIPROCITY METHOD

The MRM transforms domain integrals occurring in thermoelasticity problems into a series of boundary integrals. Obtained representations are exact forms of primary integrals since simplifications are introduced only in the stage of discretization. The MRM does not require any internal points but it does require some analytical work. This refers to the generation of subsequent Laplacians $b^{(L)}$ and $w^{(L)}$ of the source function.

Numerical implementation of the MRM is fairly simple and straightforward. It is important to note that the interchange of the summation and integration signs in (14), (26) and (27) avoids the storage of the high order fundamental solution matrices and allows to integrate directly over the boundary only once reducing the computational effort. To integrate directly, it is necessary to calculate the source term at each integration point over the element producing much more accurate results.

It is also important to point out that in many practical situations the series in (14), (26) and (27) become finite summations. Then accuracy of the MRM depends only on the boundary discretization. If source function $b^{(0)}$ generates an infinite series a careful analysis of convergence has to be carried out.

Convergence of the series depends mainly on the maximum distance between collocation points. The smaller this distance is, the easier convergence can be reached. As an example of convergence analysis, two different heat generation functions are considered in this paper.

Radial heat source

Let's consider the case of a cylinder with the internal heat sources defined as $b^{(0)} = c/R$ where c is a constant and R is the geometrical radial distance of the field point. This function produces the following series of Laplacians of temperature and their normal derivatives:

$$T^{(L)} = -\frac{cK^{(L)}}{\lambda R^{2L-1}} \tag{38}$$

$$\frac{\partial T^{(L)}}{\partial n} = \frac{c(2L-1)K^{(L)}}{\lambda R^{2L}}\frac{\partial R}{\partial n} \tag{39}$$

where

$$K^{(L)} = (2L-3)^2 K^{(L-1)} \tag{40}$$

and $K^{(1)} = 1$.

For this case, convergence analysis (details of which are given in Appendix A) indicates that the series converges only if the condition $r_{max}/R < 1$ is satisfied where r_{max} is the maximum distance from the source point to the field point.

Exponential heat source

The algorithm to calculate the Laplacians of temperature and the respective normal derivatives for the source function $b^{(0)} = c\exp(aR)$ is presented in Appendix B.

For that case, it was found that:

- for small values of the constant a all series converge rapidly;
- for large values of the exponent a, more boundary elements are necessary since the difference of $T^{(L)}$ between the outer and inner faces becomes stronger;
- the condition $r_{max}/R < 1$ has to be satisfied.

If the above conditions are not satisfied, it is necessary to divide the domain into sub-regions until the convergence criterion is fulfilled. It is also important to take into account the appropriate number of terms in the series. For more details regarding estimation of remainder series as well as other convergence problems the reader is referred to Reference 12.

NUMERICAL EXAMPLES

Hollow cylinders have often been used in the construction of nuclear reactors. In the walls of these structures, the heat is generated due to the attenuation of gamma rays and neutrons from the reactor core. In the following examples, cylinders of different geometries with different heat sources are analysed. The analysis consists of two uncoupled parts: the first one computes the temperature and heat flux distributions at any point while the second part calculates the displacements and stresses at those points. In both parts, the multiple reciprocity method is used and the results of the second part are compared with analytical solutions and/or with another BEM approach based on cells discretization. While the latest requires from the first analysis the temperature distribution over all the domain, the MRM approach needs only the temperature and its normal derivatives on the boundary. For all the examples, the material properties of the cylindrical carbon-steel pressure vessel are given by:

Thermal conductivity	$\lambda = 43.27$ W/mK
Young's modulus	$E = 207 \times 10^3$ MPa
Poisson ratio	$\nu = 0.3$
Thermal expansion	$\alpha = 1.1 \times 10^{-5}\ \mathrm{K}^{-1}$

Example 1—Hollow cylinder subject to a radial heat source

Let's consider first a simple geometry in order to compare the numerical results obtained by MRM and cells integration with the analytical solution[1].

The condition $r_{max}/R < 1$ is satisfied if a section of 45° is analysed. Domain under consideration and boundary conditions are shown in *Figure 1*. While in the MRM approach only 50 linear boundary elements are used, the cells approach also needs 200 triangular cells to discretize the domain.

The heat generation rate is given by $b^{(0)} = 10^4/R$.

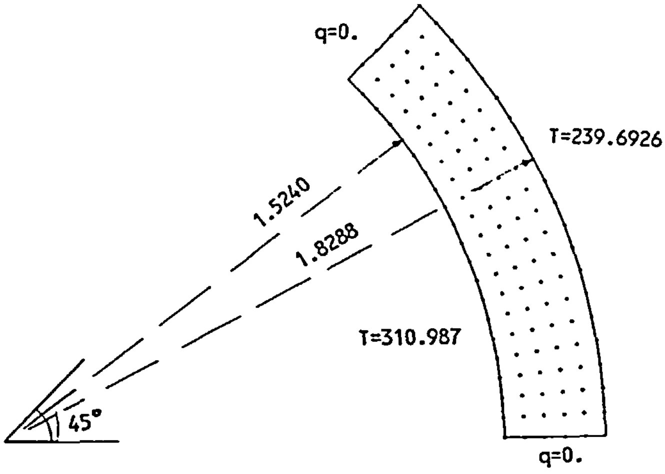

Figure 1 Geometry of the hollow cylinder and boundary conditions

Table 1 Radial displacements and stresses

	Radial displacement (cm)			σ_{zz} (MPa)		
R (m)	MRM	Cells	Analyt.	MRM	Cells	Analyt.
1.5240	0.59717	0.59711	0.59752	−744.44	−744.45	−744.04
1.5850	0.62469	0.62464	0.62504	−697.77	−697.77	−697.52
1.6459	0.65034	0.65027	0.65068	−651.25	−651.27	−651.03
1.7069	0.67420	0.67413	0.67452	−604.77	−604.75	−604.56
1.7678	0.69629	0.69622	0.69662	−558.22	−558.23	−558.11
1.8288	0.71671	0.71664	0.71703	−511.84	−511.86	−511.68

The series converge with 6 terms and the results for the radial displacements and the stresses σ_{zz} are given in *Table 1.*

Example 2—Hollow cylinder subject to an exponential heat source

In this example, the same cylinder is subjected to an exponential variation of the heat source given by $b^{(0)} = 75.08 \times 10^{14} \exp(-17.5R)$. Due to the extremely rapid variation of the heat source function in the radial direction, it is necessary to increase the number of boundary elements to obtain good results. The selected angle of the section is 5.625°. Boundary conditions are displayed in *Figure 2.*

The results for the radial displacements and the stresses σ_{zz} using 8 terms in the series are given in *Table 2.*

Example 3—Multi-bore cylinder with radial heat source

Consider the case of a multi-bore cylinder[2] with 12 spaced circular holes with internal heat generation $b^{(0)} = 2 \times 10^5/R$.

The problem is analysed using the MRM approach with 70 boundary elements and another BEM approach that requires 280 internal cells. Discretization as well as boundary conditions are shown in *Figure 3.*

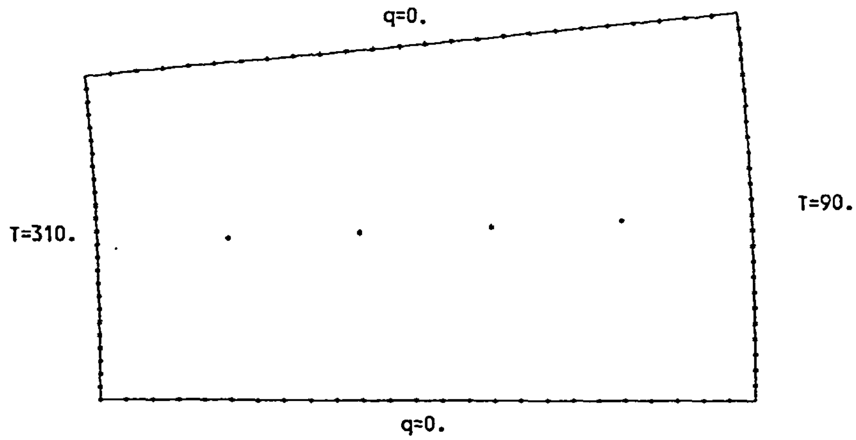

Figure 2 Boundary conditions and geometry of the hollow cylinder with the exponential heat source

Table 2 Radial displacements and stresses

	Displacement (cm)		σ_{zz} (MPa)	
R (m)	MRM	Analyt.	MRM	Analyt.
1.5240	0.42187	0.42225	−819.51	−819.31
1.5850	0.45009	0.45046	−667.74	−667.42
1.6459	0.47226	0.47261	−519.68	−519.37
1.7069	0.48878	0.48913	−376.37	−376.07
1.7678	0.50003	0.50031	−237.80	−237.56
1.8288	0.50637	0.50670	−103.80	−103.68

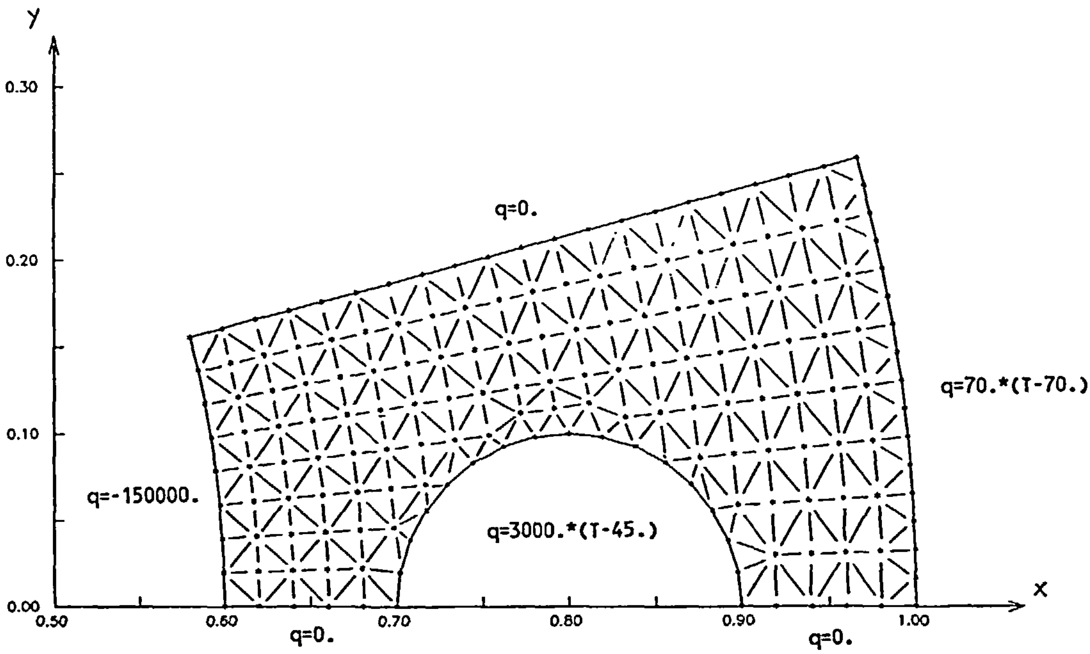

Figure 3 Discretization of the multi-bore cylinder

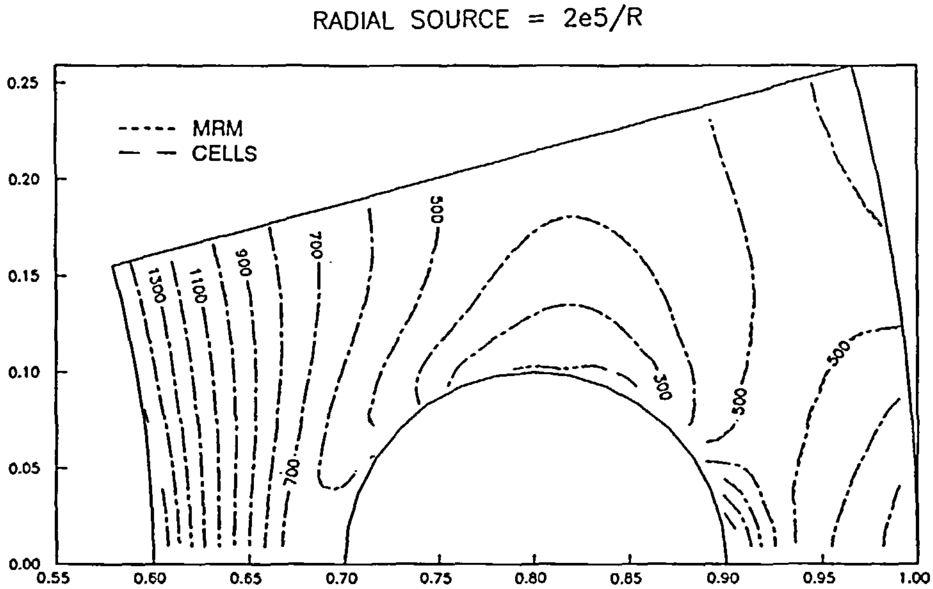

Figure 4 Contour plot of the equivalent Von Mises stresses

The resulting equivalent Von Mises stresses for both approaches are plotted in *Figure 4* and the difference was less than 1%. In this case all the series converge within 8 terms.

Example 4—Multi-bore cylinder with exponential heat source

Consider the same cylinder of the previous example but subjected to an exponential heat source $b^{(0)} = 75.08 \times 10^5 \exp(-4 \cdot R)$. Because the variation of the heat source is relatively smooth, the series converges quickly with only 8 terms.

The resulting equivalent Von Mises stresses obtained using MRM are compared with cells integration (*Figure 5*).

CONCLUSIONS

The paper presents the application of the multiple reciprocity method for solving linear thermoelasticity problems. The technique generalizes the concept of the so-called Galerkin vector and expresses solution in terms of boundary integrals only. Two independent computer codes are used. While the first code calculates the temperature and heat flux distribution along the boundary, the second one determines displacements and internal stresses.

The results of numerical tests demonstrated excellent accuracy of the MRM. It should, however, be stressed that for the very rapid variation in the heat source functions and consequently rapid variation in the temperature fields, the method requires a more refined boundary discretization and a larger number of terms in the series.

Generation of higher order Laplacians of the heat source function requires some analytical preparation. The usage of the widely available software like Derive or Mathematica, which is generally capable of automating the above process, is highly recommended. The numerical tests carried out so far showed the good efficiency of the MRM.

The reader's attention is also called to a convergence analysis.

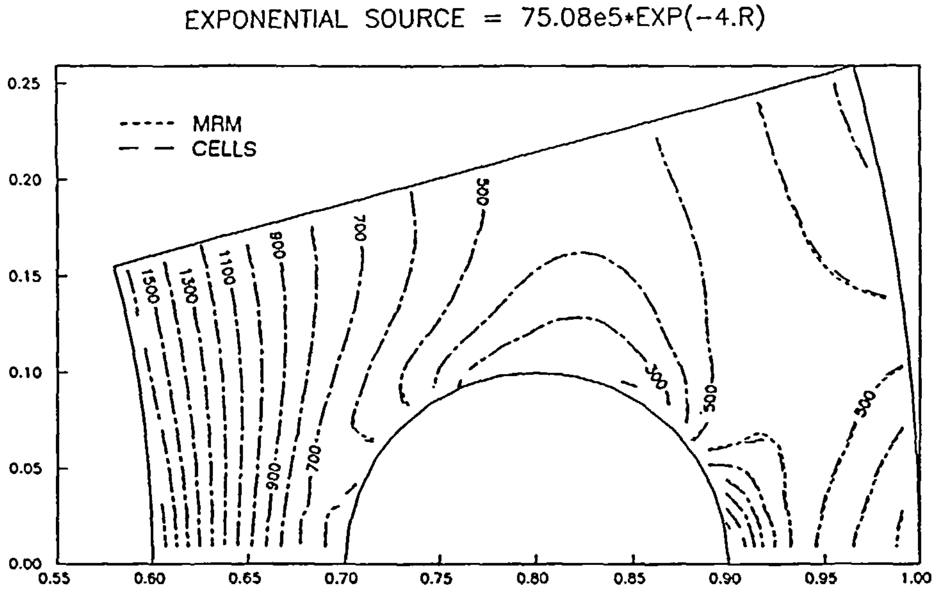

Figure 5 Contour plot of the equivalent Von Mises stresses

ACKNOWLEDGEMENT

The first author would like to thank CNPq (Brazilian Scientific Research Council) for the financial support.

REFERENCES

1 Boley, B. A. and Weiner, J. H., *Theory of Thermal Stresses*, John Wiley, New York (1960)
2 Takeuti, Y., Steady temperature distribution in a heat-generating multi-bore cylinder. *Nucl. Eng. Design*, **11**, 41–56 (1969)
3 Benjamin, M., Heat generation and temperature distributions in cylindrical reactor pressure vessels, *Nucl. Eng. Design*, **11**, 1-15 (1969)
4 Telles, J. C. F. and Brebbia, C. A., The boundary element method in plasticity, in *New Developments in Boundary Element Methods*, (Ed. C. A. Brebbia), Computational Mechanics Publications, Southampton, pp. 295–317 (1980)
5 Nowak, A. J. and Brebbia, C. A., The multiple-reciprocity method: A new approach for transforming BEM domain-integrals to the boundary, *Eng. Anal. Bound. Elements*, **6** (3), 164–168 (1989)
6 Nowak, A. J. and Brebbia, C. A., Solving Helmholtz equation by multiple reciprocity method, in *Computer and Experiments in Fluid Flow*, (Eds. G. M. Carlomagno and C. A. Brebbia), Computational Mechanics Publications, Southampton, pp. 265–270 (1989)
7 Neves, A. C. and Brebbia, C. A., The multiple reciprocity boundary element method in elasticity: A new approach for transforming domain integrals to the boundary, *Int. J. Num. Meth. Eng.*, **31**, 709–727 (1991)
8 Neves, A. C. and Brebbia, C. A., The multiple reciprocity method applied to thermoelastic problems with concentrated and distributed heat sources, in *Boundary Elements Technology VI*, (Ed. C. A. Brebbia), Computational Mechanics Publication and Elsevier Applied Science, pp. 201–214 (1991)
9 Itagaki, M., Multiple reciprocity method for one-group fission neutron source iteration problems, *Eng. Anal. Bound. Elements*, in press (1992)
10 Power, H. and Power, B. F., Solving Ossen's system of equations by boundary elements using multiple reciprocity method, *Eng. Anal. Bound. Elements*, **10**, 69–74 (1992)
11 Kamiya, N. and Andoh, E., Robust boundary element scheme for Helmholtz eigenvalue equation, in *Boundary Element Method XIII*, (Eds. C. A. Brebbia and G. S. Gipson), pp. 841–850 (1991)
12 Nowak, A. J. and Brebbia, C. A., Numerical verification of the multiple reciprocity method for linear potential problems with body forces, *Eng. Anal. Bound. Elements*, **10**, in press (1992)

APPENDIX A

Convergence of the series

Consider the heat generation $b^{(0)} = c/R$ and (28), (32), (33), (39) and (40). The series in (26) converges if for $L \gg 0$:

$$\left| u_{ij,j}^{*(L)} \frac{\partial T^{(L-1)}}{\partial n} \right| > \left| u_{ij,j}^{*(L+1)} \frac{\partial T^{(L)}}{\partial n} \right| \tag{41}$$

Consequently:

$$A^{(L)} r^{2L-1} (\ln(r) - B^{(L)}) \frac{(2L-3)}{R^{2L-2}} K^{(L-1)} > A^{(L+1)} r^{2L+1} (\ln(r) - B^{(L+1)}) \frac{(2L-1)}{R^{2L}} K^{(L)} \tag{42}$$

$$A^{(L)} \left(\frac{r}{R}\right)^{2L-2} r(\ln(r) - B^{(L)})(2L-3)K^{(L-1)} > \frac{A^{(L)}}{(2L+1)^2 - 1} \left(\frac{r}{R}\right)^{2L} r\left(\ln(r) - B^{(L)} - \frac{2(2L+1)}{(2L+1)^2 - 1}\right)(2L-1)(2L-3)^2 K^{(L-1)} \tag{43}$$

Since for $L \gg 0$ $\frac{2(2L+1)}{(2L+1)^2 - 1} \to 0$, then:

$$\frac{4L^2 + 4L}{4L^2 - 8L + 3} > \left(\frac{r}{R}\right)^2 \tag{44}$$

$$\frac{4 + \frac{4}{L}}{4 - \frac{8}{L} + \left(\frac{3}{L}\right)^2} > \left(\frac{r}{R}\right)^2 \tag{45}$$

For $L \gg 0$; $\frac{r}{R} < 1$.

APPENDIX B

Algorithm to compute the value of $T^{(L)}$ *and* $\partial T^{(L)}/\partial R$ *for the case of exponential heat source* $b = c\exp(aR)$